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Exercises for 'Functional Analysis 2' [MATH-404]

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Ex 2.1 (Topological vector spaces induced by seminorms)
Let X be a vector space equipped with a family of seminorms $(p_i)_{i\in I}$. Define a topology τ on X by setting

$$U \in \tau \iff \forall\, x \in U\ \exists I_0 \subset I \text{ finite },\ \varepsilon > 0 :\ B_{\varepsilon,I_0}(x) \subset U$$

with $B_{\varepsilon,I_0}(x) = \{y \in X : p_i(x-y) < \varepsilon \quad \forall i \in I_0\}$. Show that this notion indeed defines a topology on X and that (X,τ) becomes a topological vector space.

Solution 2.1 : We first show that τ is a topology. Clearly $\emptyset, X \in \tau$. Next, let $U_1,\dots,U_n \in \tau$ and $x \in \cap_{i=1}^n U_i$. Then there exist finite sets $I_0^i \subset I$ and $\varepsilon_i > 0$ such that $B_{\varepsilon_i,I_0^i}(x) \subset U_i$. Set $\varepsilon = \min_{1\le i\le n} \varepsilon_i > 0$ and $I_0 = \bigcup_{i=1}^n I_0^i$, so that I_0 is finite. Then $B_{\varepsilon,I_0}(x) \subset U_i$ for all $i = 1,\dots,n$ and therefore $U = \bigcap_{i=1}^n U_i$ is open. Finally, let $(U_s)_{s\in S}$ be an arbitrary family of elements in τ. If $x \in \bigcup_{s\in S} U_s$, fix $s \in S$ such that $x \in U_s$. By definition there exists $\varepsilon > 0$ and $I_0 \subset I$ finite such that $B_{\varepsilon,I_0}(x) \subset U_s \subset \bigcup_{s\in S} U_s$. Hence $\bigcup_{s\in S} U_s \in \tau$ and we conclude that τ is a topology.

Next we show that the vector space operations are continuous. Let $U \subset X$ be open and $x, y \in X$ be such that $x + y \in U$. We have to show that there exist open sets $V, W \in \tau$ with $x \in V$ and $y \in W$ such that $V + W \subset U$. Since $U \in \tau$ there exists $\varepsilon > 0$ and $I_0 \subset I$ finite such that $B_{\varepsilon,I_0}(x + y) \subset U$. We claim that $V = B_{\varepsilon/2,I_0}(x)$ and $W = B_{\varepsilon/2,I_0}(y)$ satisfy $V + W \subset B_{\varepsilon,I_0}(x + y) \subset U$. Indeed, for $x' \in V$, $y' \in W$ and $i \in I_0$ we have

$$p_i(x' + y' - (x + y)) \le p_i(x' - x) + p_i(y' - y) < \varepsilon.$$

Since a set of the form $V \times W$ is open in the product topology on $X \times X$, we conclude that the addition is continuous. To treat the scalar multiplication, let $U \in \tau$ and $\lambda x \in U$ with $\lambda \in \mathbb{R}$ and $x \in X$. Then there exists $\varepsilon > 0$ and $I_0 \subset I$ finite such that $B_{\varepsilon,I_0}(\lambda x) \subset U$. Consider $B_\delta(\lambda) \times B_{\delta,I_0}(x)$ for some $\delta > 0$ to be determined, an open neighbourhood of (λ, x) in $\mathbb{R} \times X$. For any $(\mu, y) \in B_\delta(\lambda) \times B_{\delta,I_0}(x)$, any $i \in I_0$ we have

$$\begin{aligned} p_i(\mu y - \lambda x) \le p_i(\mu y - \mu x) + p_i(\mu x - \lambda x) =& |\mu| p_i(y - x) + |\mu - \lambda| p_i(x) \\ \le& (\delta + |\lambda|)\delta + \delta p_i(x) \end{aligned}$$

If we choose $\delta = \min\{1, \frac{\varepsilon}{2}(1 + |\lambda| + \max_{i\in I_0} p_i(x))^{-1}\}$ this is less that ε, and so $\mu y \in B_{\varepsilon,I_0}(\lambda x)$ as required.

Ex 2.2 (The weak topology on a Banach space as LCTVS)
Let $(X, \|\cdot\|)$ be a Banach space (over $\mathbb{R}$). Recall that the **weak topology** on X is the coarsest topology such that all linear functionals $f : X \to \mathbb{R}$ that are continuous with respect to the norm convergence remain continuous. Show that X equipped with the weak topology becomes

a locally convex topological vector space.

Hint: Construct seminorms inducing the weak topology. A corollary of the Hahn–Banach Theorem might be useful to separate points.

Solution 2.2 : For a linear functional $f : X \to \mathbb{R}$ we define $p_f : X \to [0, +\infty)$ as $p_f(x) = |f(x)|$. By linearity of f and the fact that $|\cdot|$ is a norm on $\mathbb{R}$ we deduce that

$$\begin{aligned} p_f(\lambda x) &= |\lambda f(x)| = |\lambda||f(x)| = |\lambda| p_f(x), \\ p_f(x+y) &= |f(x) + f(y)| \leq |f(x)| + |f(y)| = p_f(x) + p_f(y). \end{aligned}$$

Hence p_f defines a semi-norm on X *[in the exam, such calculations can be summarized as "one can prove that p_f is a semi-norm"]*. Consider then τ, the topology induced by the seminorms $(p_f)_{f\in X'}$ [1] and denote by $\sigma(X, X')$ the weak topology on X. First note that by the Hahn-Banach Theorem (see Lecture 4 of the lecture notes), $p_f(x) = 0$ for all $f \in X'$ implies that $x = 0$. It thus suffices to show that $\tau = \sigma(X, X')$. Let $U \in \tau$. Then by definition for all $x \in U$ there exists $\varepsilon > 0$ and $f_1, \dots, f_n \in X'$ such that

$$\begin{aligned} U \supset \bigcap_{i=1}^{n} \{y \in X : p_{f_i}(y - x) < \varepsilon\} &= \bigcap_{i=1}^{n} \{y \in X : |f_i(y) - f_i(x)| < \varepsilon\} \\ = \bigcap_{i=1}^{n} \{y \in X : f_i(y) \in B_\varepsilon(f_i(x))\} &= \bigcap_{i=1}^{n} f_i^{-1}(B_\varepsilon(f_i(x))). \end{aligned}$$

Since the set $B_\varepsilon(f_i(x))$ is open, the continuity of f_i with respect to the weak topology implies that the set on the right hand side is an intersection of finitely many sets in $\sigma(X, X')$. It follows that U is also open in the weak topology and therefore $\tau \subset \sigma(X, X')$. Next, let $V \in \sigma(X, X')$ and $x \in V$. By definition the weak topology is generated by the sets $f^{-1}(U)$ with $U \subset \mathbb{R}$ and $f \in X'$ open. Moreover, since any open set $U \subset \mathbb{R}$ is the union of open balls (intervals are also considered as $1D$-balls here), we can equivalently generate the weak topology by sets of the form $f^{-1}(B_\varepsilon(z))$ with $z \in \mathbb{R}$ and $f \in X'$ such that $f^{-1}(\{z\}) \neq \emptyset$ and $\varepsilon > 0$. Note that for $x \in f^{-1}(\{z\})$ we can write

$$f^{-1}(B_\varepsilon(z)) = \{y \in X : f(y) \in B_\varepsilon(z)\} = \{y \in X : |f(y) - f(x)| < \varepsilon\} = B_{\varepsilon,f}(x) \in \tau.$$

Hence $\sigma(X, X') \subset \tau$ which concludes the proof.

Ex 2.3 (L^p spaces for $0 < p < 1$)

Let $(\Omega, \mathscr{A}, \mu)$ be a measure space and let $p \in (0, 1)$. Define

$$\begin{aligned} L^p(\mu) &= \Big\{ f : \Omega \to \mathbb{R} : \int_\Omega |f|^p \, d\mu < +\infty \Big\}, \\ \rho(f) &= \int_\Omega |f|^p \, d\mu. \end{aligned}$$

As usual, we identify the functions that are equal μ-almost everywhere.

a) Prove that $L^p(\mu)$ is a vector space and that $d(f, g) = \rho(f - g)$ is a translation-invariant metric on $L^p(\mu)$.
   Hint: For $p \in (0, 1)$ the estimate $(s + t)^p \leq s^p + t^p$ holds for all $s, t \geq 0$
b) Show that the topology induced by d turns $L^p(\mu)$ into a TVS.

1. X' is a standard notation for the dual space.

c) Assume that μ is the Lebesgue measure on $\Omega = \mathbb{R}$. Show that for every $\delta > 0$

$$\sup \{\rho(f) : f \in \operatorname{co}(B_\delta)\} = +\infty,$$

where $B_\delta = \{f : \rho(f) < \delta\}$ and $\operatorname{co}(B_\delta)$ is the convex hull of B_δ.
Hint: Consider for some $\lambda > 0$ the functions $g_n = \lambda\chi_{[n,n+1)}$ and certain convex combinations.

Solution 2.3 :
a) The inequality $(s+t)^p \leq s^p + t^p$, valid for all $s, t \geq 0$, yields both that $L^p(\mu)$ is a vector space and that $\rho(f+g) \leq \rho(f) + \rho(g)$. Moreover, using the properties of the Lebesgue integral we get

$$\rho(f) = 0 \iff \int_\Omega |f|^p \, d\mu = 0 \iff |f|^p = 0 \ \mu\text{-a.e.} \iff f = 0 \ \mu\text{-a.e.}.$$

Moreover, it obviously holds that for any $c \in \mathbb{R}$, $\rho(cf) = |c|^p \rho(f)$.

Employing these properties of ρ, we can check that

$$\begin{aligned} d(f,g) &= \rho(f-g) = \rho(f-h+h-g) \\ &\leq \rho(f-h) + \rho(h-g) \\ &= d(f,h) + d(h,g) \end{aligned}$$

and

$$\begin{aligned} d(g,f) &= \rho(g-f) = \rho(-1(f-g)) \\ &= |-1|^p \rho(f-g) = d(f,g) \end{aligned}$$

so d is a metric. Finally

$$d(f+h, g+h) = \rho(f+h-g-h) = \rho(f-g) = d(f,g),$$

so d is translation invariant.
b) Because $L^p(\mu)$ is a metric space with d, it suffices to work with sequences. Let $\rho(f_n - f) \to 0$, $\rho(g_n - g) \to 0$, and $c_n \to c$ in $\mathbb{R}$. Then

$$d(f_n + g_n, f+g) = \rho(f_n - f + g_n - g) \leq \rho(f_n - f) + \rho(g_n - g) \to 0 \tag{1}$$

and

$$\begin{aligned} d(c_n f_n, cf) &= \rho(c_n f_n - cf) \\ &\leq \rho(c_n f_n - c_n f) + \rho(c_n f - cf) = |c_n|^p \rho(f_n - f) + |c_n - c|^p \rho(f) \to 0, \end{aligned}$$

which shows that both addition and scalar multiplication are continuous.
c) Fix $\delta' < \delta$ and consider functions $g_n = (\delta')^{1/p}\chi_{[n,n+1)}$, $n \in \mathbb{N}$, where $\chi_{[n,n+1]}$ denotes the characteristic function of $[n, n+1]$. Then

$$\rho(g_n) = \int_\mathbb{R} \delta' \chi_{[n,n+1)} \, dx = \delta' \int_n^{n+1} \, dx = \delta' < \delta,$$

so $g_n \in B_\delta$ for any $n \in \mathbb{N}$. Take

$$f_n = \sum_{k=1}^n \frac{1}{n} g_k \in \operatorname{co}(B_\delta).$$

Because f_n is positive and the functions g_k and $g_{k'}$ have disjoint supports when $k \neq k'$, we get

$$\rho(f_n) = \int_{\mathbb{R}} \Big(\sum_{k=1}^{n} \frac{1}{n} g_k \Big)^p \, dx = \int_{\mathbb{R}} \sum_{k=1}^{n} \frac{1}{n^p} |g_k|^p \, dx = \delta' \sum_{k=1}^{n} \frac{1}{n^p} = \delta' n^{1-p} \to +\infty$$

as $n \to +\infty$ since $p \in (0,1)$.

Ex 2.4 (LCTVS with countable family of seminorms is metrizable)
Let X be a LCTVS with the topology defined by a countable family of seminorms $(p_n)_{n\in\mathbb{N}}$.

a) Consider the function $f(a) = a/(1+a)$, $a \geq 0$. Show that

$$f(a) \leq f(a+b) \leq f(a) + f(b).$$

for all $b \geq 0$.

b) Show that

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x-y)}{1+p_n(x-y)}$$

is a translation-invariant metric on X and the balls in this metric are balanced.

Hint: To demonstrate various properties of d it is convenient to prove instead the respective properties of the function $d_0(x) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x)}{1+p_n(x)}$, and use the identity $d(x,y) = d_0(x-y)$.

c) Verify that the metric topology induced by d is the same as the topology defined by the seminorms $(p_n)_{n\geq 1}$.

d) Show that

$$d_1(x,y) = \sum_{n=1}^{\infty} \min\left\{2^{-n}, p_n(x-y)\right\}$$

is likewise a translation-invariant metric defining the same topology.

Solution 2.4 :
a) The first inequality follows since $f(a) = 1 - 1/(1+a)$ and $1/(1+a)$ is decreasing. We use this fact together with the formula $f(a)/a = 1/(1+a)$, for $a > 0$, to infer that

$$f(a)/a \geq f(a+b)/(a+b), \quad f(b)/b \geq f(a+b)/(a+b), \quad a, b > 0.$$

Hence, altogether, $f(a) + f(b) \geq f(a+b)(a+b)/(a+b)$. In the case when a or b is zero, the desired estimate is trivial.
b) That

$$d_0(x) \geq 0 \quad \text{and} \quad d_0(x) = 0 \iff x = 0$$

follows directly from the non-negativity of seminorms and their separation property. Because $d_0(x) = \sum_n 2^{-n} f(p_n(x))$, where f is the function from part a), we obtain

$$\begin{aligned} d_0(x+y) &= \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x+y)}{1+p_n(x+y)} \leq \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x)+p_n(y)}{1+p_n(x)+p_n(y)} \\ &\leq \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x)}{1+p_n(x)} + \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(y)}{1+p_n(y)} = d_0(x) + d_0(y) \end{aligned}$$

The two above properties of d_0 yield that d is a metric (which is clearly translation invariant).

To show that d-balls are balanced, note that for all $0 < |\lambda| \leq 1$

$$d_0(\lambda x) = \sum_{n=1}^{\infty} 2^{-n} \frac{|\lambda| p_n(x)}{1 + |\lambda| p_n(x)} = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x)}{1/|\lambda| + p_n(x)} \leq \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x)}{1 + p_n(x)} = d_0(x).$$

c) We denote $B_r := \{x : \; d_0(x) < r\}$, an open d-ball centered at 0 with radius $r > 0$, and $B_{1,\dots,n;\delta} := \{x : p_k(x) < \delta, k = 1, \dots, n\}$ ($n \in \mathbb{N}$, $\delta > 0$), an open neighborhood of 0 generated by the family of seminorms $(p_k)_{k=1}^n$.
First fix n and $\delta > 0$. For $\varepsilon \in (0,1)$ we have

$$\begin{aligned} d_0(x) < \varepsilon 2^{-n} &\Rightarrow 2^{-k} \frac{p_k(x)}{1 + p_k(x)} \leq \varepsilon 2^{-n} \quad \text{for all } k \leq n \\ &\Rightarrow \frac{p_k(x)}{1 + p_k(x)} \leq \varepsilon \quad \text{for } k \leq n \\ &\Rightarrow p_k(x) \leq \frac{\varepsilon}{1 - \varepsilon} \quad \text{for } k \leq n. \end{aligned}$$

Therefore choosing ε such that $\varepsilon/(1-\varepsilon) < \delta$ and setting $r = \varepsilon 2^{-n}$ yields $B_r \subset B_{1,\dots,n;\delta}$.
Now fix $r > 0$ and n such that $2^{-n} < r/2$. Then for $x \in B_{1,\dots,n;r/2}$ we have $p_k(x) < r/2$ for all $k \leq n$ and so

$$d_0(x) = \sum_{k=1}^{n} 2^{-k} \frac{p_k(x)}{1 + p_k(x)} + \sum_{k=n+1}^{\infty} 2^{-k} \frac{p_k(x)}{1 + p_k(x)} < \frac{r}{2} \sum_{k=1}^{n} 2^{-k} + \frac{1}{2^n} < r$$

Thus for n as above, $B_{1,\dots,n;r/2} \subset B_r$.

d) To show that d_1 is a translation-invariant metric, we can argue as done for d. In particular, we only need to verify that

$$\min\{2^{-n}, p_n(x+y)\} \leq \min\{2^{-n}, p_n(x)\} + \min\{2^{-n}, p_n(y)\}.$$

Indeed, we first notice that $\min\{2^{-n}, p_n(x+y)\} \leq \min\{2^{-n}, p_n(x) + p_n(y)\}$. Then, if $p_n(x) + p_n(y) \leq 2^{-n}$ we necessarily have $p_n(x) \leq 2^{-n}$ and $p_n(y) \leq 2^{-n}$, meaning that the claim above is verified. If $p_n(x) + p_n(y) > 2^{-n}$, we only need to verify that $\min\{2^{-n}, p_n(x)\} + \min\{2^{-n}, p_n(y)\} \geq 2^{-n}$. But the worst possible case is when the minimum is achieved at $p_n(x)$ and $p_n(y)$, where however we know that $p_n(x) + p_n(y) > 2^{-n}$, thus proving the claim.
As before, let $B_r = \{x : \; d_1(x,0) < r\}$ denote an open ball in d_1 metric, and $B_{1,\dots,n;\delta}$ an open neighborhood of 0 generated by the the seminorms p_k.

First fix n and δ. Note that if there exist $k \leq n$ for which $p_k(x) > 2^{-k}$ then $d_1(x,0) \geq 2^{-k} \geq 2^{-n}$. Therefore, if we take $r < \min\{2^{-n}, \delta\}$ and assume that $x \in B_r$, we get

$$d_1(x,0) = \sum_{k=1}^{n} p_k(x) + \sum_{k=n+1}^{\infty} \min\left\{2^{-k}, p_k(x)\right\} < r < \delta$$

so it must follow that $p_k(x) < \delta$ for all $k \leq n$. Thus $x \in B_{1,\dots,n;\delta}$.

Now fix $r > 0$ and take n so large that $2^{-n} < r/2$. For x such that $p_k(x) < \delta := 2^{-n} r/2$, for all $k \leq n$, we get

$$d_1(x,0) \leq \sum_{k=1}^{n} p_k(x) + \sum_{k=n+1}^{\infty} \min\left\{2^{-k}, p_k(x)\right\} < \frac{r}{2} \sum_{k=1}^{n} 2^{-n} + \frac{1}{2^n} < r.$$

Ex 2.5 (Two counterexamples)

a) **A metric-vector space but not TVS**
Consider the plane $\mathbb{R}^2$ with the "Washington" metric

$$d(x,y) = \begin{cases} \|x-y\| & \text{if } x \text{ and } y \text{ are colinear,} \\ \|x\| + \|y\| & \text{otherwise.} \end{cases}$$

Show that scalar multiplication is continuous, but addition is not even separately continuous in this metric.

b) **Balls in metrizable LCTVS may be non-convex**
Consider $C(\mathbb{R})$ with a countable family of seminorms

$$p_n(f) = \sup\{|f(x)| : x \in [-n,n]\}, \quad n \in \mathbb{N},$$

and an induced translation-invariant metric given by

$$d(f,g) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(f-g)}{1+p_n(f-g)}.$$

Define

$$f(x) = \max\{0, 1-|x|\}, \quad g(x) = 100f(x-2), \quad h(x) = \frac{1}{2}\big(f(x)+g(x)\big),$$

and show that
$$d(f,0) = \frac{1}{2}, \quad d(g,0) = \frac{50}{101}, \quad d(h,0) = \frac{1}{6} + \frac{50}{102}.$$
Hence the ball $B(0,\frac{1}{2})$ is not convex.
Remark: One can show that $B(0,r)$ is not convex for any $0 < r < 1$.

Solution 2.5 :
a) Note that, roughly speaking, $d(x,y)$ is the distance you must travel to get from x to y when you are only allowed to move radially. The name refers to a street plan of Washington, D.C. Consider $x_n \xrightarrow{d} x$. Then, by the definition of d :

$$x = 0 \Rightarrow \|x_n\| \to 0$$

$$x \neq 0 \Rightarrow x_n, x \text{ are colinear from some point on and } \|x_n - x\| \to 0,$$

where $\|\cdot\|$ is the standard Euclidean norm on $\mathbb{R}^2$.

First, let us check that scalar multiplication is continuous. To this end assume that $r_n \to r$ in $\mathbb{R}$ and that $x_n \xrightarrow{d} x$ in $\mathbb{R}^2$. If $x = 0$, we know that $\|x_n\| \to 0$, so necessarily $\|r_n x_n\| \to 0$. Therefore $d(r_n x_n, rx) = \|r_n x_n\| \to 0$. If $x \neq 0$, the vectors x_n and x must become colinear from some point on (together with $\|x_n - x\| \to 0$) so the vectors $r_n x_n$ and rx are also colinear. Therefore, for all n large enough, $d(r_n x_n, rx) = \|r_n x_n - rx\| \to 0$ by the continuity of scalar multiplication in the Euclidean norm.

To show that addition is not continuous, fix $x \neq 0$ and take any sequence $(y_n) \subset \mathbb{R}^2$ such that $\|y_n\| \to 0$ (so $d(y_n, 0) \to 0$) and whose elements are not colinear with x. Then we know that $x + y_n$ and x are not colinear either and therefore $d(x+y_n, x) = \|x+y_n\| + \|x\| \to 2\|x\| \neq 0$.

b) For any $n_0 \in \{0,1,2,\dots\}$ and $a > 0$ consider the function

$$f_{n_0,a}(x) = af(x - n_0)$$

where $f(x) = \max\{0, 1 - |x|\}$. Then $\lambda f_{n_0,a} = f_{n_0,\lambda a}$, for any $\lambda > 0$, and

$$d(f_{n_0,a}, 0) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(f_{n_0,a})}{1 + p_n(f_{n_0,a})} = \sum_{n=n_0 \vee 1}^{\infty} 2^{-n} \frac{a}{1+a} = \frac{1}{2^{(n_0 \vee 1)-1}} \cdot \phi(a)$$

where $n_0 \vee 1 = \max\{n_0, 1\}$ and $\phi(a) = a/(1+a)$. To compute $d(f, 0)$ and $d(g, 0)$ note that $f = f_{0,1}$ and $g = f_{2,100}$.

Next, for $0 < a < b$, let us consider the function

$$h_{a,b} = \frac{1}{2} f_{0,a} + \frac{1}{2} f_{2,b} = f_{0,a/2} + f_{2,b/2}.$$

Because $a < b$ we get $p_1(h_{a,b}) = a/2$ and $p_n(h_{a,b}) = b/2$ for $n \geq 2$. Therefore

$$d(h_{a,b}, 0) = \frac{1}{2}\phi\big(a/2\big) + \frac{1}{2}\phi\big(b/2\big).$$

Since $h = h_{1,100}$ we can immediately obtain $d(h, 0)$.
Note for the remark that $B(0, \frac{1}{2})$ is not convex that in a topological vector space, the closure of a convex set is convex.